

# A quantum dot implementation of the quantum NAND algorithm

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We propose a physical implementation of the quantum NAND tree evaluation algorithm. Our approach, based on continuous time quantum walks, uses the wave interference of a single electron in a hierarchical set of tunnel coupled quantum dots. We find that the query complexity of the NAND tree evaluation does not suffer strongly from disorder and dephasing, nor is it directly limited by temperature or restricted dimensionality for 2-d structures. Finally, we suggest a potential application of this algorithm to the efficient determination of high-order correlation functions of complex quantum systems.

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Recently a new quantum algorithm was discovered that evaluates a binary tree of incommensurate conditions via the logical NAND gate in a time faster than the provably best classical algorithm [1, 2, 3, 4, 5]. One construction of this algorithm relies upon a continuous time quantum walk between coupled bound states, where the “answer” to the posed query is determined by the reflection or transmission of a single-particle wave-packet through an idealized, complex scattering device [2, 6]. Mesoscopic transport in semiconductor heterostructures have ideal properties for creating a device to implement a continuous time quantum walk. Their long phase coherence times and lengths for electronic transport allows for novel studies of the electronic properties of small-scale structures [7, 8]. Modern fabrication techniques allow for complex structures with robust controls and fast manipulation [9, 10, 11, 12], and the key features of the continuous time quantum walk approach (a Fano-like interference effect with a non-trivial set of coupled bound states) have been experimentally demonstrated [13].

In this paper we propose an approach to implementing NAND trees by mimicking the logical structure of the Hamiltonian formulation with a physical arrangement of quantum dots. Using a recursive Green’s function approach inspired by Ref. 2, we find the quantum speed-up of the quantum NAND algorithm—that it takes a time  $O(\sqrt{N})$  for  $N$  input bits—is maintained if the phase decoherence rate  $1/\tau_\phi$  [7] and the disorder (detunings and tunnel couplings) be lower than  $t/\sqrt{N}$ , where  $t$  is the characteristic strength of interdot tunnel coupling and  $N$  is the number of bits in the input to the NAND tree. One key feature of the present work is the use of a large physical space ( $O(N)$  quantum dots are used) to allow for efficient coupling to the complex quantum system (the “oracle”) to which the query is posed. We develop appropriate techniques such that temperature and restricted dimensionality lead to at most a  $O(\log N)$  overhead in the evaluation of the algorithm. We conclude with a discussion of the “quantum-ness” of the algorithm and consider how such a device can be used to measure many-

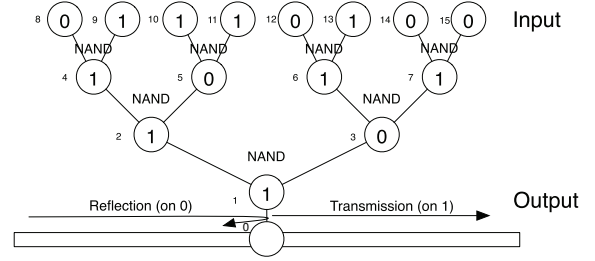


FIG. 1: The logical NAND tree with  $N = 8$  inputs in the top of the tree. Nodes are labeled from 1 to 15. Each node can be realized as a quantum dot, with the links between nodes represented by tunnel coupling between linked dots. Detunings  $b_i$  applied to dots  $i + N$  at the “top” of the tree set the local potential to be off of the Coulomb blockade peak for dot  $i + N$  if  $b_i \neq 0$ . The conductance of the dot  $i = 0$  at the bottom of the tree, near zero bias ( $E = 0$ ), encodes the result of the NAND operation.

body properties of complex quantum systems.

*Quantum dot implementation:* The NAND tree quantum walk is described by the transmission coefficient of a series of coupled states, shown in Fig. 1. Each “state”  $|i\rangle$  in the graph corresponds to the lowest orbital state of a quantum dot in the Coulomb blockade regime, with  $i = 1$  labeling the bottom quantum dot at level  $k = 0$  and  $i = j + 2^k - 1$  labeling the  $j$ th dot at the  $k$ th level. We assume a large orbital level spacing  $\alpha \gg t$ , where  $t$  is the average tunnel coupling in the tree of dots. The coupled dot Hamiltonian can be written:

$$H = \sum_i \epsilon_i |i\rangle \langle i| - t_{2i} |i\rangle \langle 2i| - t_{2i+1} |i\rangle \langle 2i+1| + \text{H.c.} \quad (1)$$

where  $|i\rangle = c_i^\dagger |n_i\rangle$  describes the addition of a (spin-less) electron on the  $i$ th dot near the charge transition from  $n_i \rightarrow n_i + 1$ . Each dot can have a slight detuning from the Fermi energy of  $\epsilon_i$ , and each dot  $i$  is tunnel coupled to dots  $2i$  and  $2i+1$  with a tunnel coupling  $t_{2i}, t_{2i+1} \sim t$ , forming a physical binary tree structure.

At the top of the tree is an oracle ( $\mathcal{O}$ ), a quantum

system with electron number operators  $\hat{n}_{i\mathcal{O}}$  coupled to the binary tree via  $V = \Delta \sum_{i=0}^{N-1} (-)^i \hat{n}_{i\mathcal{O}} |i + N\rangle\langle i + N|$ . Unlike previous work, we focus on a coupling that may be easy to implement physically: for example, the oracle could be a mesoscopic device capacitively coupled to the top of the tree. We will proceed with the case of an oracle in an eigenstate of the  $\{\hat{n}_{i\mathcal{O}}\}$  operators with eigenvalues  $\{b_i\}$  each taking values 0 or 1. This sets the detuning of dots at the top of the tree to  $\epsilon_{i+N} = (-)^i b_i \Delta$ .

The algorithm progresses by the injection of an electron wavepacket into the bottom of the tree. This probes the Green's function of the bottom quantum dot, which depends recursively on the Green's functions for the left and right subtrees. For wavepackets with small energy spreads  $\Delta E < t/\sqrt{N}$ , the successive, coherent scattering events up and down the tree lead to either total transmission or total reflection of the wavepacket, where the result (transmission or reflection) is the final "answer" to the NAND query for an input specified by the bits  $b_i$ . A voltage probe in the linear response regime provides such an energy probe for the system: the conductance for electrons passing through the bottom of the tree encodes the result of the tree. To show that our approach works in the presence of dephasing and disorder, we develop a recursive description of the scattering Green's function in the quantum dot tree in the spirit of previous work [2].

*Green's function approach:* We now describe an approach to understanding the NAND tree evaluation using a recursively defined Green's function for single-particle propagation through the tree. If we consider a "tree" segment with a single state, 1, at the bottom, we can define the projectors  $P_1, Q_l, Q_r$ , where  $Q_l$  projects into the subspace associated with the left side of the tree (all states connected to dot 2) and  $Q_r$  similarly projects into the subspace associated with the right side. We can write the inverse Green's function exactly [14]:

$$\begin{aligned} G_1^{-1}(E) &= E + i\gamma - P_1 H P_1 - \\ &\sum_{j=l,r} P_1 H Q_j \frac{1}{E + i\gamma - Q_j H Q_j} Q_j H P_1 \\ &= E + i\gamma - \epsilon_1 - |t_l|^2 G_l(E) - |t_r|^2 G_r(E) \end{aligned} \quad (2)$$

This gives a recursive relation for the projected Green's function  $G_i(E)$  for a state  $i$  in terms of the Green's functions for the subtrees connected to states  $l = 2i$  and  $r = 2i + 1$ . The term  $\gamma = 1/\tau_\phi$  is a measure of the charge-based dephasing the electron may be expected to encounter.

At the top (largest fanout) of the tree, we can write the single-state Green's function  $G_i^{\text{top}}(E) = \frac{1}{E + i\gamma - \epsilon_i}$  which truncates the recursion relation. We remark that this recursion is only successful because the tree contains no loops; otherwise, Eq. (2) would fail. In the disorder-free

case,  $t_i = t$  for all  $i$  and  $\epsilon_i = 0$  except for dots at the top of the tree with input bits  $b_k \neq 0$ , as described above.

Setting energy to units of  $t$ , we consider the Green's function at the top of the tree for the three different inputs. We set  $s_i$  to be the sign of the detuning to illuminate the role the alternating sign  $(-)^i$  plays. We assume  $\gamma, E \ll 1 \lesssim \Delta$ . For dot  $j$ , tunnel coupled to  $2j$  and  $2j+1$ , we get (to  $O(t/\Delta, \gamma/t)$ ):

$$G_j(E) = \begin{cases} -E/2 - i\gamma/2 & b_{2j} = 0, b_{2j+1} = 0 \\ -E - i\gamma & b_{2j} = 0, b_{2j+1} = 1 \\ \frac{1}{E + i\gamma + \frac{s_{2j} + s_{2j+1}}{\Delta}} & b_{2j} = 1, b_{2j+1} = 1 \end{cases} \quad (3)$$

As the  $s_i$  alternate such that each paired "1" input has  $s_{2j} = -s_{2j+1}$ , the third term has no energy shift. We now identify Green's functions of the forms

$$G_{\text{"0"}} = \frac{1}{\alpha E + i\gamma\beta}, \quad G_{\text{"1"}} = -(\alpha E + i\gamma\beta) \quad (4)$$

where the "0" and "1" indicate the expected NAND results for the inputs. Any dependence on  $\Delta$  is removed by the alternation of  $s_i$  for paired "1" inputs.

We can include effect of disorder in the system, both in variations in  $t_i$  and in variations in  $\epsilon_i$ . Both are likely to occur due to differences in fabrication, low frequency noise on electrostatic gates (such as those used to tune the system to the multi-dot regime) and charge-fluctuators in the system. For the three different inputs (00,01,11) at the top of the tree:

$$\begin{aligned} 0,0 \rightarrow & -E \frac{(\epsilon_l - i\gamma)^2 t_l^2 + (\epsilon_r - i\gamma)^2 t_r^2}{[\epsilon_r t_l^2 + \epsilon_l t_r^2 - i\gamma(t_l^2 + t_r^2)]^2} \\ & + \frac{1}{\frac{t_l^2}{\epsilon_l - i\gamma} + \frac{t_r^2}{\epsilon_r - i\gamma}} \end{aligned} \quad (5)$$

$$1,0 \rightarrow -E \frac{1}{t_r^2} + \frac{\epsilon_r - i\gamma}{t_r^2} \quad (6)$$

$$1,1 \rightarrow \frac{1}{E + (i\gamma + (s_l t_l^2 + s_r t_r^2)/\Delta - \epsilon_j)} \quad (7)$$

These have the same form as the disorder free results of Ref. 2 so long as  $t_l, t_r \geq t$ , where  $t$  is now the minimum tunnel coupling. More generally, we will assume  $t_i$  and  $\epsilon_i$  to be drawn from a normal distribution with means  $t, 0$  and standard deviations  $\sigma_t, \sigma_\epsilon$ . We can remove any explicit  $\Delta$  dependence as long as  $\sigma_t/\Delta \ll 1$ .

We now show that two subtree's described by Green's functions of the above type lead to the appropriate type Green's function at the joining of the two subtrees, recursively implementing the NAND tree. Consider the general map for two subtrees' Green's functions  $G_{l(=2j)}$  and  $G_{r(=2j+1)}$ , parameterized by  $\alpha_{l,r}, \beta_{l,r}$ .  $G_j$  for the cases 00, 10, and 11 are given to  $O(\gamma/t, \epsilon_j/t, t/\Delta)$ :

$$\frac{1}{\alpha_l E + i\gamma\beta_l}, \frac{1}{\alpha_r E + i\gamma\beta_r} \rightarrow - \left[ \frac{\alpha_l t_r^2 \beta_r^2 + \alpha_r t_l^2 \beta_l^2}{(t_r^2 \beta_l + t_l^2 \beta_r)^2} E + i\gamma \frac{\beta_l \beta_r}{t_r^2 \beta_l + t_l^2 \beta_r} \right] \quad (8)$$

$$-(\alpha_l E + i\gamma\beta_l), \frac{1}{\alpha_r E + i\gamma\beta_r} \rightarrow - \left[ \frac{\alpha_r}{t_r^2} E + i\gamma \frac{\beta_r}{t_r^2} \right] \quad (9)$$

$$-(\alpha_l E + i\gamma\beta_l), -(\alpha_r E + i\gamma\beta_r) \rightarrow \frac{1}{(1 + t_l^2 \alpha_l + t_r^2 \alpha_r) E + i\gamma(1 + t_l^2 \beta_l + t_r^2 \beta_r + i\epsilon_j/\gamma)} \quad (10)$$

The overall map implements a NAND gate by identification of “0” and “1” via Eq. 4.

For a tree consisting entirely of the “most dangerous case” (an input of the form 1011) subtrees of Ref. 2, we find  $\alpha \rightarrow 2^{k/2}$ , and  $\beta \rightarrow 2^{k/2} + i \sum_j \epsilon_j / \gamma$ , where the sum is over the set of all dots where both subtrees evaluate to 1. The first term indicates that the Green’s function derived is only appropriate within an energy range  $t/2^{n/2} = t/\sqrt{N}$  about  $E = 0$ . This leads to the condition that the electron wavepacket be spread over a time  $\sim \sqrt{N}/t$ , and sets  $\sqrt{N}$  scaling of the evaluation time. For random, uncorrelated noise, the second term (with  $O(N)$  terms) has a root-mean-square of  $\sigma_\epsilon \sqrt{N}$ , and leads to an energy shift by that amount from the desired resonance at  $E = 0$  as well as a dephasing  $\gamma 2^{n/2}$ . This indicates that  $\gamma, \sigma_\epsilon \ll t/\sqrt{N}$  form the requirements for operation of the device. Less difficult is the bounding requirement  $t_i \geq t$ . We compared these results with numerical simulations and find consistent behavior for tree sizes up to  $N = 128$ . A sample with  $N = 32$  is shown in Fig. 2.

*Measuring with a quantum dot:* We now detail how a quantum dot coupled to a linear voltage probe can measure the result of the NAND tree well below the temperature limit of the leads. This is a generalization of sub-temperature spectroscopy used in double quantum dot transport measurements [9].

We write the Green’s function, including the self energies  $\Sigma_{l,r}^R$  for left and right leads coupled to the dot via  $t_{l,r}$ , as

$$G_0^{-1}(E) = E - \epsilon_0 - \Sigma_l^R - \Sigma_r^R - |t_1|^2 G_1(E) \quad (11)$$

By working well above the band-bottom of the leads, we can write  $\Gamma = \Gamma_l + \Gamma_r$  with  $\Gamma_{l,r} = 2t_{l,r}^2 \nu = -2i\Sigma_{l,r}^R$ , where  $\nu$  is the density of states near the Fermi energy. The Meir formalism [15] for the transmission  $\bar{T}_{lr}(E) = \text{Tr}[\Gamma_l G_{l,0} \Gamma_r G_{l,0}^\dagger]$  yields

$$\bar{T}_{lr}(E) = \frac{\Gamma^2/4}{(t_1^2 \text{Im}[G_1(E)] - \Gamma/2)^2 + (E - t_1^2 \text{Re}[G_1(E)] - \epsilon_0)^2} \quad (12)$$

The conductance follows from the Landauer formula,  $G_{lr} = e^2/h \int f'(E - E_f) \bar{T}_{lr}(E) dE$ , where  $f(E)$  is the Fermi distribution.

A few remarks are in order. First, a pole at  $E = 0$  for  $G_1(E)$  leads to zero conductance, while a “1”-like

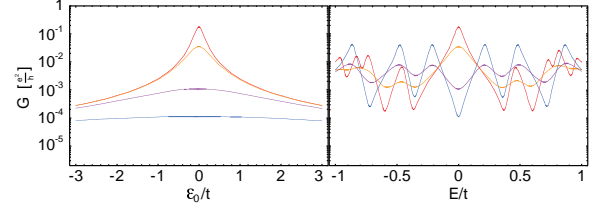


FIG. 2: Simulations for  $N = 32$  with outputs “0” (blue,purple) and “1” (red,orange). The dependence of conductance on  $\epsilon_0$  (left) and  $E$  (right) are shown. Disorder parameters  $\gamma = \sigma_\epsilon = \sigma_t = 0.03t$  (red,blue) and  $0.1t$  (orange,purple) are used.

Green’s function (proportional to energy), leads to a peak at  $\epsilon_0 = 0$ , thus letting the existence or absence of transport indicate a “1” or “0” result. Second, when dephasing ( $\gamma$ ) is weak, the transmission function is a Lorentzian with a width at most  $\Gamma/2$ . When convolved with the derivative of the Fermi function at  $k_b T > \Gamma$ , the edges of the Fermi function are effectively damped, leading to sub-temperature energy resolution (limited only by  $\Gamma$ ). To see these effects, we simulate the expected conductance as a function of  $\epsilon_0$  and  $E$  at  $T = 0$  in Fig. 2 for an input of  $N = 32$  bits. Cases resulting in “0” and “1” are both shown. For this scenario, we set  $\Gamma/t = 0.1 < 2^{-5/2}$  and  $\Delta/t = 10 > 2^{5/2}$ . The temperature dependence can be mitigated by further reductions of  $\Gamma$ . However, disorder, as described below, may limit the achievable resolution.

*Two-dimensional layout:* One difficulty encountered for making large NAND trees using this approach is the increasing perimeter of the tree as a function of tree depth ( $N$  versus  $\log N$ ). For sufficiently large trees, the spacing between successive levels  $k$  and  $k+1$  of the tree must grow as  $2^{k/(d-1)}$  for a  $d$ -dimensional layout. Furthermore, tunnel coupling exponentially decreases as the spacing between quantum dots increases, potentially limiting this approach to small trees.

As an alternative, “dummy” levels of the tree can be inserted: these are states coupled only to a single dot going up or down the tree (Fig. 3a). Each acts as an inverter, where the “missing” state is equivalent to the absence of a resonance in one “leg” of the tree, which is the “1”-type Green’s function. Specifically, a state of

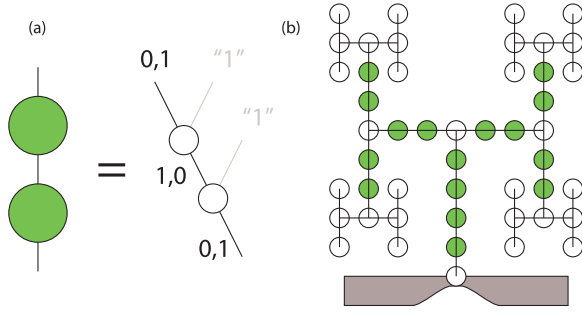


FIG. 3: (a) An inline quantum dot act as an “inverter”—the non-existent right inputs to each dot act as a virtual “1”. Two inline dots move the bit in the top input to the bottom output. (b) An array of tunnel coupled quantum dots in an H-fractal layout, with pairs of inverters connecting more distant quantum dots. The number of inverters between dots doubles every *two* levels of the tree.

either “0” or “1” type parameterized by  $(\alpha, \beta)$  becomes, after coupling through two inverters,  $(1+\alpha, 1+\beta)$ . Going a distance  $d$  (through  $2d$  inverters) gives

$$(d + \alpha, d + \beta) \quad (13)$$

As an explicit construction, we consider the H-fractal (Fig. 3b) for the layout of quantum dots. The H-fractal is a binary tree structure comprised of an “H” with the two sides the same length as the middle bar. [16] This has the benefit that the distance between successive levels of the binary tree doubles for every *two* levels. A “worst case” trees (again, with inputs 1011) has the mapping

$$(\alpha_{k+2}, \beta_{k+2}) = (2^{k/2} + 2\alpha, 2^{k/2} + 2\beta) \quad (14)$$

Where the distance goes as  $2^{k/2}$  at the  $k$ th level (with  $n$  the bottom of the tree). A tree comprised entirely of worst-case trees has

$$\alpha_n = \beta_n = 2^{n/2} + 2(2^{n/2-1} + 2(\dots(1)\dots)) \quad (15)$$

$$< n2^{n/2} = \log_2(N)\sqrt{N} \quad (16)$$

This indicates that a 2D architecture leads to an overhead of  $\log(N)$  for the evaluation of the tree. This still performs better than the best classical algorithm. As a final remark, we can selectively remove one of the pair of dots to produce a “NOT” operation inline with the following NAND operation by opening the tunnel coupling between the pair of dots. This allows the tree evaluator to evaluate *arbitrary* boolean functions, not just NAND trees.

*Practical limits:* Taking values  $\gamma = 0.1\mu\text{eV}$ ,  $t = 100\mu\text{eV}$ ,  $\alpha = 1\text{ meV}$ ,  $\Gamma = 0.1\mu\text{eV}$ ,  $\sigma_\epsilon = \sigma_t = 1\mu\text{eV}$ , and  $k_B T = 2\mu\text{eV}$  (20 mK dilution refrigerator electron temperature), we find the limiting algorithm size  $N \sim 2^{13}$ , where the limit is due to the disorder in the system. Using the H-fractal layout, this corresponds to a mesoscopic

region of area  $a \times 3^{13} \lesssim 0.02\text{ mm}^2$  for dot center-to-dot center spacing  $a \sim 100\text{ nm}$ . This includes appropriate space for electrostatic gates or etched barriers. Phase coherent transport over such large regions have been demonstrated in a number of experiments [8]. The expected evaluation time is of order  $10/\Gamma \lesssim 60\text{ ns}$  (to generate a differential signal of ten electrons, sufficient for fast detection).

We can use this NAND tree evaluator as a subsystem in a larger processing unit. The best known classical algorithm evaluates in  $N^{0.753}$  time. Using the quantum device as a sub-tree evaluator of a quantum oracle, a problem of size  $2^{13+k}$  can be evaluated in  $2^{6.5+0.753k} < 2^{9.79+0.753k}$  time, as subtrees of size  $\leq 2^{13}$  can be calculated sequentially using the quantum subsystem as a hardware accelerator. Better hybrid quantum-classical approaches may exist, but require further investigation.

*Conclusion:* This paper illustrates a fundamental and heretofore unrecognized quality of the quantum NAND algorithm. In particular, the entire NAND quantum walk can be done with a single-electron wavepacket and appropriate resonances (quantum dots). Other approaches using classical waves, such as light in coupled cavities, would in principle have the same Green’s function. Thus, from a quantum information perspective, the interesting part of the algorithm lies in the oracle: the quantum mechanical system which “sets” the inputs. We heretofore considered the oracle to be in an eigenstate of the number operators  $\hat{n}_{i\mathcal{O}}$ . If we describe the boolean function evaluation of the tree by  $f(b_0, \dots, b_{N-1}) = 0$  or  $1$ , an arbitrary oracle state leads to an expected result of the tree of  $\langle f(\hat{n}_{0\mathcal{O}}, \dots, \hat{n}_{N-1\mathcal{O}}) \rangle_{\mathcal{O}}$ . For example, a two-level tree evaluates  $\langle \hat{n}_{0\mathcal{O}}\hat{n}_{1\mathcal{O}} + \hat{n}_{2\mathcal{O}}\hat{n}_{3\mathcal{O}} - \hat{n}_{0\mathcal{O}}\hat{n}_{1\mathcal{O}}\hat{n}_{2\mathcal{O}}\hat{n}_{3\mathcal{O}} \rangle$ , a non-trivial, high-order correlation function of the oracle. We now have the outstanding task of identifying interesting quantum systems to use as an oracle.

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